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Parametric Design and Optimisation of Thin-Walled Structures for Food Packaging

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Abstract. In this paper the parametric design and functional optimisation of thin-walled structures made from plastics for food packaging is considered. These objects are produced in such vast numbers each year that one important task in the design of these objects is to minimise the amount of plastic used, subject to functional constraints, to reduce the costs of production and to conserve raw materials. By means of performing an automated optimisation on the possible shapes of the food containers, where the geometry is parameterised succinctly, a strategy to create the optimal design of the containers subject to a given set of functional constraints is demonstrated.

Keywords: Structural Optimisation, Parametric Design, Thin-Walled Structures

1. Introduction

There is no doubt that the recent progress in the performance of computing hardware has made available sophisticated numerical methods for real time engineering design and analysis. Integrated Computer-Aided Design (CAD), mesh generation, and finite element analysis have placed the process of engineering design at a very rational level as opposed to a 'cut-and-try' approach practised for many years (Farouki, 1999). Although still not particularly common today, the integration of numerical optimisation techniques as part of the Computer-Aided Engineering (CAE) process is an avenue with potential for further fruitful exploration. For a given product, the task of automating the process of CAE and optimisation can be thought of being able to perform the various steps of the design process, (namely model creation by means of a CAD technique, the relevant physical analyses and more importantly being able to optimise the shape) with a minimum amount of user intervention. As pointed out by many authors, e.g. Amara *et al* (Amara *et al* , 1997) and Farouki (Farouki, 1999), the major difficulty in automating the process of CAE is the proper linking of the various stages of the design process. For this reason it is desirable to have a mechanism which enables us to consider the various stages of the design process in a systematic manner.

The aim of this paper is to describe a strategy for automatic design optimisation within an interactive environment, by means of physical analysis, of thin-walled structures made from polymeric materials. In particular the optimal design of such structures by means of creating a parameterised CAD model is discussed. By making use of an optimisation routine, automatic variations of the parameters associated with the CAD model enable to create a rich variety of possible designs which are then subjected to physical analysis. The process is terminated when the parameters defining a design which fits our requirement is found. Thus, the process by which automatic design optimisation is made feasible is shown and, furthermore, it is shown that the key for this is the efficient and consistent parameterisation of the geometry model.

For the purpose of demonstrating the techniques, the task considered here is to minimise the amount of material used in packaging subject to a given set of functional constraints such as the strength and the volume, in order to reduce the cost of production and to conserve the raw materials used. The potential impact of this research comes from the vast range of products used in the food industry composed of thin-walled polymers. For example, the types of products that make use of such containers are dairy (yoghurt and ice-cream containers, mayonnaise and cooking sauces), convenience foods (ready meals and snack foods), clear packs (sandwich packaging, wines and beers). Thus, one can readily appreciate that one of the challenges facing the relevant manufacturing industries at present is to reduce the amount of material within a container, whilst at the same time maintaining adequate strength and volume. Therefore, it is of no doubt that given the cost of raw materials, the ability to reduce the amount of material used to fabricate a particular object, whilst maintaining its required physical properties such as strength, would have a significant economic and environmental impact. For example, a saving of 1gm of material from a typical margarine tub might, for one factory over the space of a year, save 48 tons of raw material or about £33,000 ($46,760).

In the food packaging sector, the nature of product development often leads to very short time scales. Many food products undergo some changes to their packaging on an annual basis, and some much more frequently than this. These may be purely cosmetic (for increased customer impact, re-branding etc) or be more fundamental in nature. e.g. reduction of weight. A typical request of this latter type would be '*make pack X 10% lighter but maintain rigidity*'. For a typical manufacturer of food containers the target for producing samples which demonstrate the new concept is usually within 4 working days. To achieve this they must design/redesign the product, have a sample mould made and then produce the sample containers. It is clear that during this process, the length of time for radical product design and more importantly optimisation using conventional approaches is currently either small or non-existent. Thus, it is worthwhile to address the issue of automatic design optimisation within the context of functional design of thin-walled structures used for food packaging.

Generally speaking, a typical requirement in practical engineering design is to minimize or maximize a measure of merit (an objective function) without violating a set of constraints. The mathematical formulation of such an optimisation problem can be written as:

$$\min\{\ f(\underline{x})\ \mid\ \underline{x}_l \le \underline{x} \le \underline{x}_u;\ \ \underline{g}(\underline{x}) = 0;\ \ \underline{h}(\underline{x}) \le 0\ \},\quad \underline{x} \in \Re^n, \tag{1}$$

with

f the objective function;
$\underline{x}$ vector of n design variables;
$\underline{g}$ vector of p equality constraints;
$\underline{h}$ vector of q inequality constraints;
$\underline{x}_l$ and $\underline{x}_u$ lower and upper bounds for the design variables.

The design variables and the constraints form the feasible design space

$$\underline{x} \in R^n\ \mid\ \underline{x}_l \le \underline{x} \le \underline{x}_u;\ \ \underline{g}(\underline{x}) = 0;\ \ \underline{h}(\underline{x}) \le 0, \tag{2}$$

in which the trial design should lie.

It has been noted by several authors in the shape optimisation literature that the most important aspect of shape optimisation is the choice of the design variables to be used and how the shape is parameterised in terms of these design variables (Francavilla et al., 1975; Imam, 1982; Yao and Choi, 1989; Yoo et al., 1984). Choosing too many variables will considerably complicate the design problem with severe implications for the computational time required, and having too few variables may result in only a limited range of design alternatives being obtained (Francavilla et al., 1975; Imam, 1982). It is therefore a basic requirement that a wide range of shapes (which can be defined by a relatively small number of parameters) are accessible to the method of optimisation used.

Applying the idea of parameterisation schemes to structural optimisation in packaging is not entirely new. For example, Bhakuni *et al* (Bhakuni et al. , 1991) performed optimisation on beverage cans where the main objective was to minimise the weight of the can. The weight of the can in this case was measured as the diameter of the blank sheet used to create the can. More recently, Dijk *et al* (Dijk et al., 2000) describe optimal design of bottles with ribs by trying to minimise the mean thickness subject to the shape of the ribs. In this case the design parameters were those introduced on the ribs allowing the shape of the ribs to be changed. However, a major drawback in these optimisation problems is the lack of an efficient and coherent mechanism to parameterise the entire geometry of the object under consideration.

In this paper it is shown how the automatic design optimisation of thin-walled structures can be carried out by efficiently defining and parameterising the shape of these structures. For this purpose the PDE method (Bloor and Wilson , 1989) is used. The method adopts a boundary-value approach in which the shape of the object in question is decomposed into a series of surface patches bounded by 'character-lines', where the number of patches are being kept as low as possible. Using the boundary data appropriately defined along the character-lines the PDE method enables to produce smooth surfaces between them. More importantly, the shape of the surface produced is efficiently parameterised with minimal number of shape parameters thus enabling automatic optimisation to be carried out.

In what follows the geometry model of the shape of the food container is used to generate a valid finite element mesh with which to analyse its physical properties, in particular its load-bearing characteristics. The results of the finite element computation provide with a measure for the strength of the container. The aim here is to identify a shape of a container which uses as little material as possible whilst possessing a predefined level of strength. In order to carry out a realistic optimisation a volume constraint in the form of an equality condition is also introduced within the optimisation procedure. Then using standard optimisation procedures, the shape parameters, defining the geometry of the container, are iteratively modified to improve the object's load-bearing characteristics for a fixed volume.

For the work described here the PDE chosen is of the form:

$$\left(\frac{\partial^2}{\partial u^2} + a^2\frac{\partial^2}{\partial v^2}\right)^2 \underline{X}(u,v) = 0, \tag{3}$$

where conditions on the function $\underline{X}(u,v)$ and its normal derivatives $\frac{\partial \underline{X}}{\partial n}$ can be imposed at the edges of the surface patch. The parameter a is a special design parameter which controls the relative smoothing of the surface in the u and v directions (Bloor and Wilson , 1990). For periodic boundary conditions (e.g. $0 \leq u \leq 1$, $0 \leq v \leq 2\pi$), a pseudo-spectral method has been developed for the solution of equation (3) which allows $\underline{X}(u,v)$ to be expressed in closed form (Bloor and Wilson , 1996). This solution method is briefly outlined in the appendix **A**.

As far as parametric design is concerned, one way of defining the boundary conditions are in terms of curves in 3-space. For example,

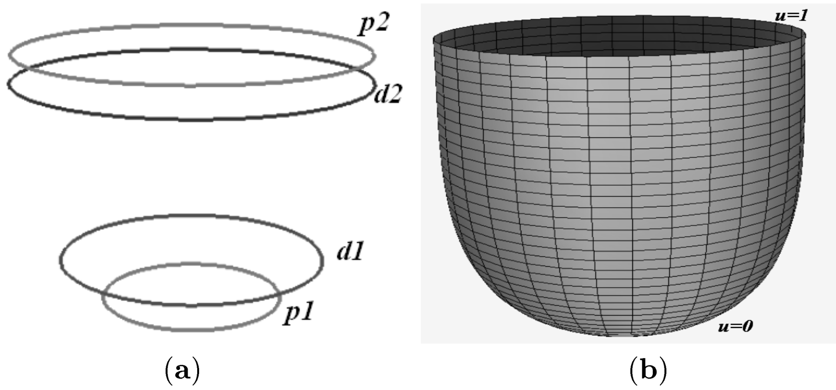


Figure 1. The PDE surface corresponding to the bowl of a container. (**a**) The boundary curves. (**b**) The corresponding PDE surface patch.

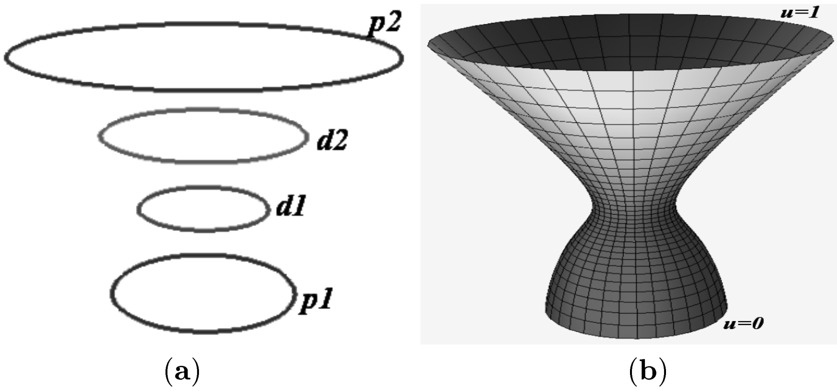
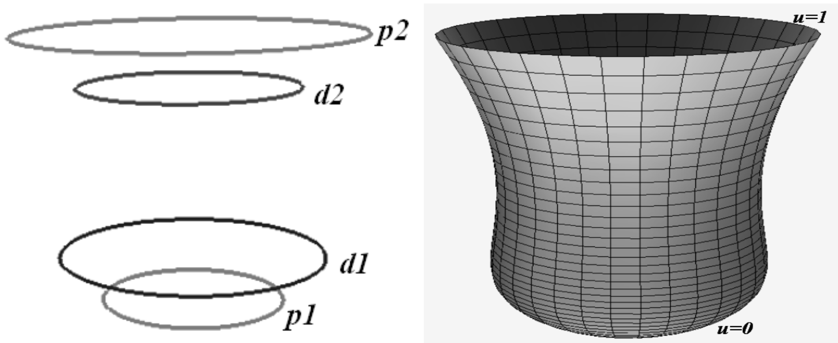


Figure 2. The effect on the shape of the surface by interactively manipulating the derivative boundary curves for the surface shown in Figure 1. (**a**) The boundary curves. (**b**) The corresponding PDE surface patch.

Figure 1 shows a typical set of boundary curves and the corresponding PDE surface corresponding to the bowl portion of a yoghurt container. Here the value of a was taken to be 1.01. Note that the curves marked $p1$ and $p2$ correspond to the boundary conditions on the function $\underline{X}(u, v)$, where $p1(v) = \underline{X}(0, v)$ and $p2(v) = \underline{X}(1, v)$. A vector field corresponding to the difference between the points on the curves marked $p1$ and $p2$ and those marked $d1$ and $d2$ respectively, corresponds to the conditions on the function $\frac{\partial \underline{X}}{\partial n}$ such that:

$$\frac{\partial \underline{X}}{\partial n} = \left[\underline{p}(v) - \underline{d}(v)\right] s, \tag{4}$$

where s is a scalar. The conditions defined by $p1$, $p2$ and $d1$, $d2$ are known as the 'positional boundary conditions' and 'derivative boundary conditions' respectively (Ugail et al., 1999a). Note that the surface



(**a**) (**b**)

Figure 3. The effect on the shape of the surface by changing the design parameters corresponding to the boundary $d = 2$ for the surface shown in Figure 1. (**a**) The boundary curves. (**b**) The corresponding PDE surface patch.

patch will not necessarily pass through the curves which define the derivative boundary conditions. The surface shown in Figure 2 illustrates the effect of changing the derivative condition, where the derivative boundary curves $d1$ and $d2$ shown in Fig. 1(**a**) have been interactively resized and vertically translated away from the corresponding position boundary curves.

1.1. Parameterisation of the Geometry Model

In this work the definition of the shape geometry is based on the interactive PDE parameter model discussed in Ugail *et al* (Ugail et al., 1999b), where the parameterised boundary curves are used to define the shape of the surface. Essentially, this parameterisation is defined in such a way that linear transformations, such as translation, rotation and dilation, of the boundary curves can be carried out interactively. The result of this is that the designer is presented with tools which enable him/her to create and modify the geometry in an intuitive manner (Ugail et al., 1999a; Ugail et al., 1999b).

For the work presented here, the parameterisation on the boundary curves is denoted using the notation c_{kP_i} $(k = 1, 2)$, $(i = x, y, z)$. Here c indicates the type of curve, with the letter p denoting the position curves and the letter d denoting the derivative curves. The index k ranges from 1 to 2 corresponding to the $u = 0$ and $u = 1$ boundary edges (respectively) of the surface. The letter P denotes the type of parameter: T for a translation, R for a rotation and D for a dilation. Finally the letter i denotes the coordinate directions relevant to a particular type of parameter. Adjustments to the values of these

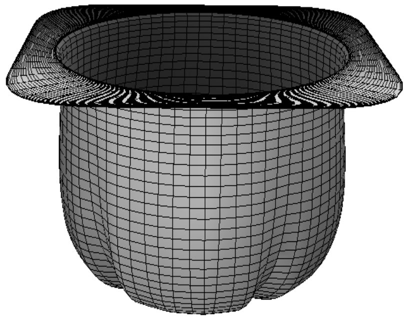
Table I. Values for the design parameters for the boundary $d = 2$ of the surfaces shown in Fig. 1(**b**) and Fig. 3(**b**).

Parameter	Figure 1	Figure 3
d_{2T_x}	0.000	0.000
d_{2T_y}	0.850	0.750
d_{2T_z}	0.000	0.000
d_{2D_x}	0.700	0.500
d_{2D_y}	0.700	0.500
d_{2D_z}	0.000	0.000
d_{2R_x}	0.000	0.000
d_{2R_y}	0.000	0.000
d_{2R_z}	0.000	0.000

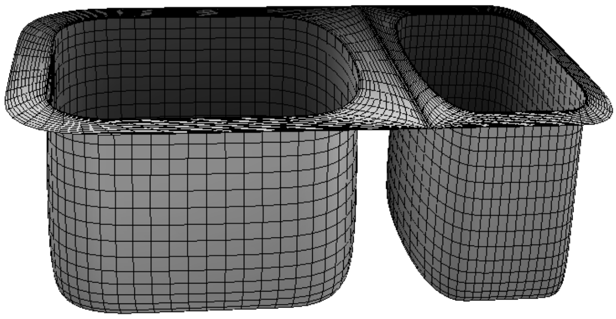
parameters along with the value of a in equation (3) can be used to create and manipulate complex geometries.

As mentioned earlier, the effect of these parameters on the surface shape is easy to appreciate. Table I shows the values of the chosen parameters for $d = 2$ for the surface shown in Figure 1. In order to show the effect of the design parameters we now choose a different set of values for the parameters for the boundary $d = 2$ of the surface shown in Figure 1. The new values chosen for the parameters are shown in Table I and the resulting surface is shown in Figure 3. Note the value of a for the surface shown in Figure 3 is the same as that shown in Figure 1, i.e. a=1.01. Essentially, the new values of the parameters produced a dilation followed by a translation of the boundary curve $d = 2$. The parameters introduced on the boundary curves are varied using a graphical interface where the corresponding surface is visualized simultaneously. The spectral approximation method for solving the PDE, mentioned earlier, is fast enough for the surfaces to be created and manipulated in real time.

In order to create the geometry corresponding to complex shapes, more than one surface patch often needs to be joined together along common boundaries thereby enabling one to form a composite surface (Ugail et al., 1999a; Ugail et al., 1998). The parametric model discussed above has been extended to cater for such composite bodies. For example, Fig. 4(**a**) shows the shape of a yoghurt container created using two surface patches. The ridges at the base of the container are created so as to obtain a more realistic shape for the container. This is



(**a**)



(**b**)

Figure 4. Composite shapes build from multiple patches. (**a**) Shape of a yoghurt container with a single compartment. (**b**) Shape of a yoghurt container with dual compartments.

done by means of the corresponding derivative curve which in this case is defined by means of a cubic B-Spline (Schumaker, 1981). Details of how this is done are discussed later in this paper. Similarly, Fig. 4(**b**) shows the shape of a yoghurt container with dual compartments created using 4 separate surface patches. In each case the composite shapes are parameterised by means of the boundary curves corresponding to the surface patches which make up the shape.

2. Design Optimisation

This section shows how automatic design optimisation of thin-walled containers can be carried out using the PDE parametric model discussed above. In particular, two examples taken from a practical setting is discussed, i.e., the optimisation for strength of two yoghurt containers with single and dual compartments shown in Fig. 4(**a**) and Fig. 4(**b**) respectively.

As mentioned earlier, a typical design optimisation problem can be thought of as maximizing or minimizing an objective function without violating a set of constraints. There exist a wide variety of methods for numerical optimisation. The choice of a particular method is problem specific and involves considerations such as the computational cost of evaluating the function to be optimised, and also the behaviour of the function within the design space.

Here the optimisation is performed by solving a constrained optimisation problem formulated by means of the objective function f, the design parameters associated with the geometry of the shape and the constraints imposed. i.e. strength and volume. This is carried out using an augmented Lagrange multiplier method (Greig, 1980) along with the Broyden-Fletcher-Goldfarb-Shanno (BFGS) method (Vanderplaats, 1984; Press et al., 1992). The BFGS method is a Quasi-Newton which performs a series of local minimisations of the objective function f along a straight line in the parameter space. The method is iterative where at each successive iteration we obtain a vector $\underline{x}^k = ({x_1}^k, {x_2}^k, ..., x_n^k)$ of the n independent design variables which is computed from the previous iterations using the expression:

$$\underline{x}^{k+1} = \underline{x}^k + \alpha^k \underline{s}^k. \tag{5}$$

Here $\underline{s}^k$ is a direction of search and α^k is a scalar that minimises the one-dimensional function $F(\alpha) \equiv f(\underline{x}^k + \alpha^k \underline{s}^k)$. Thus, given a starting point, the algorithm moves in a series of steps through points in the parameter space giving a lower value of the objective function than previously, until it finds a local minimal value of the objective function. The augmented Lagrange multiplier method enables one to solve the optimisation problem with equality conditions. The BFGS method is chosen here for the sake of illustration. Other constrained optimisation methods such as Sequential Quadratic Programming (SQP) could equally be used, although the number of function evaluation is a consideration, since the cost of each finite element analysis is not negligible.

Since the BFGS is a gradient based optimisation scheme, during the optimisation we need to perform a design sensitivity analysis so that the necessary gradients are fed to the minimisation algorithm. Here we have chosen to evaluate the design sensitivity numerically by means of a finite difference scheme. In particular, the gradients of the objective function f are approximated by forward difference.

2.1. Calculation of the Objective Function

Here we are concerned in the design of containers possessing the minimum amount of material subject to a required strength. Moreover, the problem of stacking the containers on top of each other for the purpose of transportation and display on the supermarket shelves is considered. Assuming we consider a container at the bottom of a stack, such a container experiences a stress (due to the weight of the rest of the containers in the stack) and hence becomes slightly deformed. It is the excessive shear stress which can cause most damage to the material under consideration. Thus, a measure for the required strength of the container can be computed by calculating the maximum shear stress within the container. This is done by means of thin-shell finite element analysis where a force is applied around the rim of the container which translates to the tension in the top seal of the container due the weight of the other containers in the stack. It is also assumed that the base of the container is fixed.

Given the PDE geometry corresponding to the shape of a container, this geometry is then discretised, by means of the 2-dimensional (u, v) parameter space, to obtain a valid finite element mesh. In particular, the task of node numbering is relatively easy to deal with using the discretised (u, v) parameter space. To create the appropriate shell elements a thickness is generated by means of calculating normals to the surface points defining the finite element mesh. For the yoghurt containers discussed here a mean thickness distribution of 0.8mm throughout the containers is assumed. This assumption is taken to simplify the problem, since industrial forming processes do not often produce containers with a constant thickness distribution. As far as the material properties are concerned, assuming the material under consideration is isotropic and linearly elastic, the Young's modulus and Poisson's ratio to be that of polystyrene, i.e. 2.4GPa and 0.33 respectively. Figure 5 shows the displacement profile predicted by the finite element analysis for a typical loading for the stacking problem discussed above, where a tension of $15\mathrm{Nm}^{-1}$ was applied to the top seal of the container.

As mentioned above, the design objective here is the minimisation of the mass of the container subject to a given maximum shear stress. Hence the process of optimisation requires the calculation of the maximum shear stress that occurs in the solution for every design to be analysed. To do this The maximum shear stress σ^p_{max} occurring on any plane through a point p is first calculated. Using this the measure for the strength of the container to be the maximum shear stress occurring

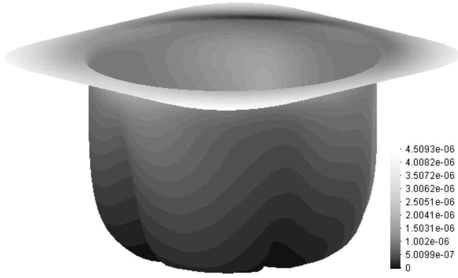


Figure 5. The displacement profile of the containter due to stacking, predicted by the finite element analysis.

in the whole structure, i.e.

$$f = \underbrace{\max}_{\text{(all points)}} \{\sigma^p_{\max}\}. \tag{6}$$

is calculated.

2.2. Optimisation of a Yoghurt container with a Single Compartment

Here the automatic optimisation of the container shown in Fig. 4(**a**) is considered. This particular container is created using two PDE surface patches (one for the bowl and the other for the flat flange). The objective here is to determine the optimal shape corresponding to the bowl part of the container which is parameterised by means of the shape defining boundary curves discussed earlier. Since we are interested in determining the internal shape of the container, changes in the parameters introduced on the derivative boundary curves of the bowl part of the container is only considered. In particular the translation in y direction and dilations in the xy plane of these two curves within defined limits is considered to obtain a favourable range of shapes. The ridges introduced at the base are created so as to obtain a realistic shape for the container, where such ridges are commonly found in food containers for a variety of reasons. These ridges are created by means of the derivative boundary curve corresponding to the base of the container. This curve is defined using a cubic B-spline definition such that:

$$\underline{d}_1(v) = \sum_i \underline{c}_i B_i(v), \tag{7}$$

where B_i is a cubic B-spline, and $\underline{c}_i$ are the control points.

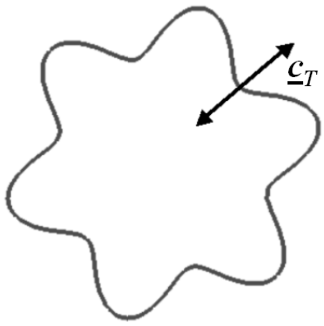


Figure 6. Cubic B-spline curve corresponding derivative condition enabling to create the ridges at the base of the container.

The control points, $\underline{c}_i$, of the spline are chosen so as they initially lie on the curve which determine the value of dilation parameter. The number of control points chosen determine the number of ridges on the container. In order to define the amplitude of the ridges we translate the control points, normal to the curve, within a confined xy planar region. Thus, the amount of translation of the control points, normal to the curve determine the prominence of the ridges. The translation of the control points, introduces an extra shape parameter which is referred as $\underline{c}_T$. Figure 6 shows the B-spline curve illustrating the parameter $\underline{c}_T$.

With the above formulation the design parameters and their initial values for the optimisation of the yoghurt container are shown in Table II. Note that the table also shows the chosen range for each parameter. The range specified for each design parameter (by means of interactively choosing a maximum and a minimum) allows the parameters to be varied within the specified ranges enabling alternative shapes to be created within the design space automatically. These ranges are chosen to ensure that the geometry of sensible shapes are fed into the optimisation routine. The container was loaded so as a tension of 15Nm^{-1} was applied to the top seal of the container. The required strength of the container is specified so as $\sigma^p_{\max} \leq$ 15MPa. The design space is further restricted by choosing a volume constraint for the container. For this particular example the volume of the container is fixed to 150ml.

Once the geometry is parameterised, the design parameters and their ranges along with the value for the required volume of the container are fed into the optimisation routine. This routine then automatically searches the design space in order to find the design with lowest possible value of the chosen merit function. Due to the extensive finite element analysis calculations, needed to calculate the value of the objective

Table II. Parameter values for the yoghurt container with a single compartment.

Parameter	Minimum	Maximum	Initial	Optimal
d_{1T_y}	-0.400	-0.001	-0.400	-0.133
d_{1D_x}	0.100	0.800	0.450	0.298
d_{1D_y}	0.100	0.800	0.450	0.309
d_{2T_y}	0.001	0.400	0.400	0.396
d_{2D_x}	0.100	0.800	0.450	0.371
d_{2D_y}	0.100	0.800	0.450	0.379
a	1.000	7.000	1.000	1.095
$\underline{c}_T$	-0.300	0.300	0.200	0.002

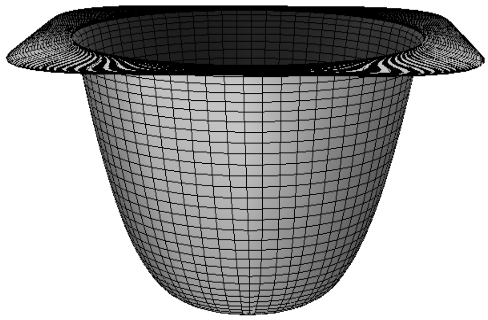


Figure 7. Optimal design for the yoghurt container with a single compartment.

function for each iteration of the optimisation, it took a little over 24 hours for the optimisation to complete on a PC workstation with a 800 MHz processor.

The values of the parameters obtained for the optimal design is shown in Table II and the optimal shape is shown in Figure 7. The resulting optimal shape had a relative reduction in mass of 29.5%. The maximum shear stress occurring within the resulting optimal shape was found to be 14.31MPa. As of note is that the optimal design shows a narrower base and the ridges introduced at the base have also vanished.

2.3. Optimisation of a Yoghurt container with Dual Compartments

As with the previous example of the container with single compartment, here the optimal design of the container with dual compartments are considered. This particular shape of the container is created by means of

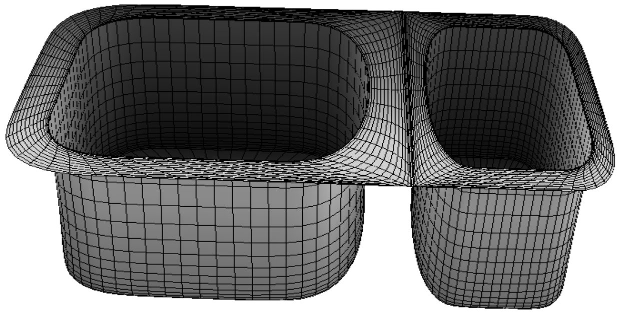


Figure 8. Optimal design for the yoghurt container with dual compartments.

4 individual surface patches. Again the aim here to predict the internal shape of the container, the variation of the parameters introduced into the internal shape of the container is only considered. In particular the translation in y direction and dilations in the xy plane of the derivative boundary curves, along with the variation of the smoothing parameter a for both compartments are considered. Unlike the previous example, in order to minimise the number of design parameters the ridges at the base of the container is not considered. It is important to note that compared to the previous example, the number of parameters and the number of elements in the finite elements mesh are almost doubled.

The parameter values for the optimisation of the yoghurt container is shown in Table III and Table IV, where the parameterisation for larger compartment is shown in Table III and that for the smaller compartment is shown in Table IV. Again the ranges for the parameters are chosen to ensure that the geometry of sensible shapes are fed into the optimisation routine. This time the container was loaded so as a tension of $15\mathrm{Nm}^{-1}$ and $10\mathrm{Nm}^{-1}$ was applied to the top seal of the container of larger and smaller compartments respectively. The required strength of the container was again specified so as $\sigma^p_{\mathrm{max}} \leq 25\mathrm{MPa}$. With the above parameterisation a volume constraint of 250ml and 125ml are imposed on the larger and smaller compartment respectively. With this formulation the automatic optimisation was performed in the hope of finding a shape possessing the required characteristic. For this particular case, with the computing hardware mentioned earlier, the optimisation took almost 48 hours to complete.

The values of the parameters obtained for the optimal design is shown in Table III and IV. The shape of the container corresponding to these parameters is shown in Figure 8. The resulting optimal shape

Table III. Parameter values for the larger compartment of the yoghurt container with dual compartments.

Parameter	Minimum	Maximum	Initial	Optimal
d_{1T_y}	-0.400	-0.001	-0.400	-0.398
d_{1D_x}	0.100	0.250	0.210	0.241
d_{1D_y}	0.100	0.250	0.210	0.239
d_{2T_y}	0.001	0.400	0.400	0.390
d_{2D_x}	0.100	0.250	0.210	0.220
d_{2D_y}	0.100	0.250	0.210	0.218
a	1.000	7.000	1.000	1.122

Table IV. Parameter values for the smaller compartment of the yoghurt container with dual compartments.

Parameter	Minimum	Maximum	Initial	Optimal
d_{1T_y}	-0.400	-0.001	-0.400	-0.380
d_{1D_x}	0.100	0.250	0.210	0.240
d_{1D_y}	0.100	0.250	0.210	0.242
d_{2T_y}	0.001	0.400	0.400	0.392
d_{2D_x}	0.100	0.250	0.210	0.218
d_{2D_y}	0.100	0.250	0.210	0.215
a	1.000	7.000	1.000	1.101

had a relative reduction in mass of 11%. The maximum shear stress occurring within resulting optimal shape was found to be 23.16MPa.

3. Conclusion

In this paper it has been demonstrated how the PDE method for the description and parameterisation of complex geometries can be used to setup the shape optimisation of practical thin-walled objects. In particular, two examples of yoghurt containers is considered demonstrating how the chosen PDE can be used to parameterise the geometry and more importantly how automatic optimisation by means of physical analysis can be performed on them. For the examples discussed here, in order to simplify the optimisation problem, the assumption of constant of thickness distribution throughout the containers has been chosen. Work is currently in progress where optimisation of similar thin-walled structures with variable thickness distribution is considered. Furthermore, the examples considered here are somewhat generic in terms of the geometry of the shapes and setting of the automatic optimisation itself. Nonetheless the concept has been clearly demonstrated where a practical automatic optimisation process can be carried out in an industrial setting.

It is noteworthy that presently when considering the physical properties of new designs made from thin-walled structures, the designers usually rely upon their own knowledge and experience, and further along the design process, model testing. However, in this paper it is shown that the geometry defining and the succinct parameterisation characteristics of the PDE method can be used to carry out automatic design optimisation in a practical setting where the time consuming and tedious process of creating trial designs and model testing can be significantly reduced.

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Appendix

A. Outline of the solution method for the PDE surface

To generate a typical PDE surface equation 3, solved over a finite region Ω of the (u, v) parameter plane subject to the boundary conditions on the solution $\underline{X}(u, v)$ which specify how $\underline{X}(u, v)$ and its normal derivatives $\frac{\partial \underline{X}}{\partial \underline{n}}$ vary along $\partial\Omega$.

With periodic boundary conditions, v being the periodic parameter, and using the method of separation of variables, the analytic solution of equation 3 can be written as,

$$\underline{X}(u, v) = \underline{A}_0(u) + \sum_{n=1}^{\infty} [\underline{A}_n(u) \cos(nv) + \underline{B}_n(u) \sin(nv)], \qquad (8)$$

where

$$\underline{A}_0 = \underline{a}_{00} + \underline{a}_{01} u + \underline{a}_{02} u^2 + \underline{a}_{03} u^3, \qquad (9)$$

$$\underline{A}_n = \underline{a}_{n1}e^{anu} + \underline{a}_{n2}e^{anu} + \underline{a}_{n3}e^{-anu} + \underline{a}_{n4}e^{-anu}, \tag{10}$$

$$\underline{B}_n = \underline{b}_{n1}e^{anu} + \underline{b}_{n2}e^{anu} + \underline{b}_{n3}e^{-anu} + \underline{b}_{n4}e^{-anu}, \tag{11}$$

where $\underline{a}_{n1}$, $\underline{a}_{n2}$, $\underline{a}_{n3}$, $\underline{a}_{n4}$, $\underline{b}_{n1}$ $\underline{b}_{n2}$, $\underline{a}_{n3}$ and $\underline{a}_{n4}$ are vector-valued constants, whose values are determined by the imposed boundary conditions at $u = 0$ and $u = 1$.

For a given set of boundary conditions, in order to define the various constants in the solution, it is necessary to Fourier analyse the boundary conditions and identify the various Fourier coefficients. For a finite number of Fourier modes (typically $N \leq 6$) the approximate surface solution can be written as,

$$\underline{X}(u,v) = \underline{A}_0(u) + \sum_{n=1}^{N}[\underline{A}_n(u)\cos(nv) + \underline{B}_n(u)\sin(nv)] + \underline{R}(u,v), \tag{12}$$

where $\underline{R}(u,v)$ is a remainder function defined as,

$$\underline{R}(u,v) = \underline{r}_1(v)e^{wu} + \underline{r}_2(v)e^{wu} + \underline{r}_3(v)e^{-wu} + \underline{r}_4(v)e^{-wu}, \tag{13}$$

where $\underline{r}_1$, $\underline{r}_2$, $\underline{r}_3$, $\underline{r}_4$ and w are obtained by considering the difference between the original boundary conditions and the boundary conditions satisfied by the function

$$\underline{F}(u,v) = \underline{A}_0(u) + \sum_{n=1}^{N}[\underline{A}_n(u)\cos(nv) + \underline{B}_n(u)\sin(nv)]. \tag{14}$$

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